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FREE CONVECTION FROM A SPHERE IN A SLIGHTLY-THERMALLY STRATIFIED FLUID

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Abstract—The steady, axisymmetric flow of a vertically stratified viscous fluid over a fixed sphere is considered in a uniform gravity field. Analytical solutions are obtained by the singular perturbation technique valid for small (modified) Grashof numbers. Two cases are considered, viz. when the sphere is either thermally insulated or when its surface temperature is maintained constant equal to that of the fluid occupying the diametral plane. Streamlines are shown graphically in an axial plane for the flow near the sphere. It is found that for the thermally insulated sphere, the flow is like the one about a stationary sphere in a rotating fluid, i.e. the inflow near the equator changes into an outflow at the poles, the transition occurring at an angle of 54.5° measured from the poles. For the isothermal sphere, the streamlines are similar to those of a uniformly spinning sphere in a fluid at rest.

NOMENCLATURE

All primed quantities are dimensional; all unprimed quantities are dimensionless. Subscripted terms with m denote their corresponding values at the diametral plane ($y' = 0$).

a', radius of the sphere;
A_n, B_n, C_n, D_n, E_n, set of arbitrary constants;
g', acceleration of gravity;
G, square root of modified Grashof number $\{g'\beta'a'^4(\mathrm{d}T'_\infty/\mathrm{d}y')\}^{1/2}/\nu'$;
J_0, J_1, Bessel function of the first kind and of order zero and one respectively;
k', thermal conductivity;
P, Prandtl number;
P_n, Legendre polynomials of the first kind and order n;
Q_l, Q_u, amount of heat absorbed by the lower and upper hemispheres respectively;
Q_v, added vertical heat flux due to the sphere;
r', r, radial coordinate $r = r'/a'$;
s, steepness parameter defined as $\{2a'(\mathrm{d}T'_\infty/\mathrm{d}y')\}$ divided by the temperature difference between the sphere and the fluid occupying the diametral plane;
T', T, temperature, $T' = T_m + a'(\mathrm{d}T'_\infty/\mathrm{d}y')T$;
$\mathrm{d}T'_\infty/\mathrm{d}y'$, constant temperature gradient, describing the constant stratification;
v'_r, v_r, r-component of velocity, $v_r = v'_r a'/\nu' G$;
v'_θ, v_θ, θ-component of velocity, $v_\theta = v'_\theta a'/\nu' G$;
y', y, $y' = r'\cos\theta$, $y = y'/a'$ ($= r'\cos\theta/a'$).

Greek symbols
α', thermal diffusivity;
β', volumetric coefficient of thermal expansion;
θ, colatitude or polar angle measured from the upward vertical $\theta = 0$;
ρ', density;
ν', kinematic viscosity;
ψ', ψ, Stokes stream function $\psi = \psi'/G\nu' a'$;
δ, $\delta = -1$ for isothermal sphere and $\delta = 0.5$ for thermally insulated sphere.

1. INTRODUCTION

FREE convection heat transfer from spheres at low Grashof numbers has recently drawn considerable interest. Analytical studies involving spherical geometries have been presented by Mahony [1], Mack and Hardee [2], Hossain and Gebhart [3] and Fendell [4]. A number of experimental investigations at small and large Grashof numbers have also been reported in the literature [5–10]. Eichhorn *et al.* [11] performed experiments on natural convection from isothermal spheres and cylinders immersed in a thermally stratified fluid and presented heat-transfer data and visual observations of the flow field. Of interest to us is their qualitative description of the behavior of laminar plumes from isothermal spheres in terms of a steepness parameter s.

The results presented in this paper correspond to the sphere problem investigated by Eichhorn *et al.* [11] at small values of G when $s = \infty$. Theoretical solutions are obtained by the singular perturbation technique up to the second power in G for the isothermal and thermally insulated sphere. Streamlines for the isothermal sphere indicate that the inflow at the poles changes into an outflow near the equator, similar to the flow for a rotating sphere in a fluid at rest. Although the photograph presented by Eichhorn *et al.* for $s = \infty$ in their Fig. 6 is true for large Grashof numbers, we find that the streamlines sketched in this paper are in qualitative agreement with their results. For thermally insulated spheres, the flow lines are similar to those on a stationary sphere in a rotating fluid (Singh [12]). It is thus shown that there exists an analogy between rotating and thermally stratified fluid flows as described by Yih [13] and Veronis [14].

2. FORMULATION

We consider an otherwise undisturbed viscous fluid having a density distribution which varies slightly in the vertical but is constant in horizontal planes. A sphere is introduced which is either thermally insulated or whose surface temperature is maintained constant equal to that of the fluid at infinity in the (horizontal) diametral plane. The Navier–Stokes equations for steady, axisymmetric motion are (see Mack and Hardee [2] and Hossain and Gebhart [3]):

$$D^4\psi = -Gr\sin\theta\left(\sin\theta\frac{\partial T}{\partial r}+\frac{\cos\theta}{r}\frac{\partial T}{\partial \theta}\right) + G\sin\theta\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}\right)\left(\frac{D^2\psi}{r^2\sin^2\theta}\right), \quad (1)$$

$$\nabla^2 T = \frac{PG}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right), \quad (2)$$

where

$$D^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta},$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

The dimensionless velocity components are related to ψ as given by the following

$$v_r = \frac{\partial\psi/\partial\theta}{r^2\sin\theta}, \quad v_\theta = \frac{-\partial\psi/\partial r}{r\sin\theta}. \quad (3)$$

In the above equations, the Boussinesq approximation has been made (see Spiegel and Veronis [15]).

$$\frac{\rho'-\rho'_m}{\rho'_m} = -\beta'(T'-T'_m), \quad T' = T'_m + a(\mathrm{d}T_\infty/\mathrm{d}y)T. \quad (4)$$

The boundary conditions are

$$\psi = \partial\psi/\partial r = 0,\ T = 0 \text{ or } \partial T/\partial r = 0 \text{ at } r = 1 \quad (5)$$

$$\nabla\psi/r \to 0, \quad T \to r\cos\theta(\text{or } y) \quad \text{as} \quad r\to\infty. \quad (6)$$

The flow depends on two parameters G (square root of modified Grashof number) and P (Prandtl number). We attempt to solve the above system (1) and (2) subject to conditions (5) and (6) in ascending powers of G [P fixed] but a uniformly valid solution for small values of G does not exist (Kaplun and Lagerstrom [16] and Proudman and Pearson [17]). The Stokes solution of (1) and (2) valid near the sphere, is obtained by satisfying the conditions (5). The Oseen approximation satisfies the conditions (6) and is true at infinity. The undetermined constants of the two solutions are evaluated by the matching technique.

3. SOLUTION

(i) *Stokes expansion*

For $r \simeq O(1)$, we assume an expansion of the form

$$\psi = \psi_0(r,\theta)+G\psi_1(r,\theta)+G^2\psi_2(r,\theta)+\ldots \quad (7)$$

$$T = T_0(r,\theta)+GT_1(r,\theta)+G^2T_2(r,\theta)+\ldots \quad (8)$$

and substitute into equations (1) and (2). Since ψ and $\partial\psi/\partial r$ vanish both at $r = 1$ and $r = \infty$, ψ_0 is zero throughout. By similar arguments T_1, $T_3,\ldots$, and ψ_2, $\psi_4,\ldots$ etc. are zero. For T_0, equation (2) gives

$$\nabla^2 T_0 = 0 \quad (9)$$

whose solution satisfying (5) and in view of (6) is

$$T_0 = A\cos\theta\left(r+\frac{\delta}{r^2}\right), \quad (10)$$

where $\delta = -1$ for isothermal sphere (i.e. $T_0 = 0$ at $r = 1$) and $\delta = 0.5$ in case of thermally insulated sphere ($\partial T_0/\partial r = 0$ at $r = 1$). For ψ_1, we obtain from (1)

$$D^4\psi_1 = 3A^2\delta\sin^2\theta\cos\theta/r^2. \quad (11)$$

Solution of (11), such that ψ_1 has a double zero at $r = 1$ is (Proudman and Pearson [17])

$$\psi_1 = \sum_1^\infty \{B_n[(2n-1)r^{n+3}-(2n+1)r^{n+1}+2r^{-n+2}] + C_n[2r^{n+1}-(2n+1)r^{-n+2}+(2n-1)r^{-n}]\}Q_n(\cos\theta) + (A^2\delta\sin^2\theta\cos\theta/8r^2)(r^2-1)^2 \quad (12)$$

where

$$Q_n(\mu) = \int_1^\mu P_{n-1}(\mu)\,\mathrm{d}\mu \quad (13)$$

and P_n are the Legendre polynomials of the first kind. The particular solution in (12) is such that $\Delta\psi_1/r$ does not vanish as r approaches infinity. And hence (12) violates the condition (6). The reason for this breakdown is that inertia and viscous terms become comparable at large values of r.

(ii) *Stretched variables and Oseen expansion*

The inertia and viscous terms far away from the sphere are of the order of $\{r'^4 g'\beta'(\mathrm{d}T'_\infty/\mathrm{d}y')/r'^3\}$ and $\{v'[r'^4 g'\beta'(\mathrm{d}T'/\mathrm{d}y')]^{1/2}/r'^3\}$ respectively. Hence their ratio is $O\{r^2[a^4 g'\beta'(\mathrm{d}T'_\infty/\mathrm{d}y')]^{1/2}/v'\}$, i.e. at infinity

$$r^2 G \simeq O(1). \quad (14)$$

Equation (14) suggests for the Oseen's variables

$$r = \rho/G^{1/2}, \quad T = x/G^{1/2}, \quad \psi = H/G^{3/2} \quad (15)$$

in terms of which the governing equations (1) and (2) become

$$D_\rho^4 H = -\rho\sin\theta\left(\sin\theta\frac{\partial x}{\partial\rho}+\frac{\cos\theta}{\rho}\frac{\partial x}{\partial\theta}\right) + \sin\theta\left(\frac{\partial H}{\partial\theta}\frac{\partial}{\partial\rho}-\frac{\partial H}{\partial\rho}\frac{\partial}{\partial\theta}\right)\left(\frac{D_\rho^2 H}{\rho^2\sin^2\theta}\right) \quad (16)$$

$$\nabla_\rho^2 x = \frac{P}{\rho^2\sin\theta}\left(\frac{\partial H}{\partial\theta}\frac{\partial x}{\partial\rho}-\frac{\partial H}{\partial\rho}\frac{\partial x}{\partial\theta}\right) \quad (17)$$

where

$$D_\rho^2 = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{\rho^2}\frac{\partial}{\partial\theta}$$

$$\nabla_\rho^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

The conditions (5) and (6) are

$$H = \partial H/\partial \rho = 0, \quad x = 0 \quad \text{or} \quad \partial x/\partial \rho = 0, \quad \text{at} \quad \rho = G^{1/2}, \tag{18}$$

$$\nabla H/\rho \to 0, \quad x \to \rho\cos\theta \quad \text{as} \quad \rho \to \infty. \tag{19}$$

The Oseen expansion is assumed as

$$x = x_0(\rho,\theta)+f_1(G)x_1(\rho,\theta)+f_2(G)x_2(\rho,\theta)+\dots \tag{20}$$

$$H = H_0(\rho,\theta)+f_1(G)H_1(\rho,\theta)+f_2(G)H_2(\rho,\theta)+\dots \tag{21}$$

such that $f_{n+1}(G) = O\{f_n(G)\}$, $n = 1,2,\dots$. The leading terms of the Oseen solution, on account of (19) are

$$H_0 = 0, \quad x_0 = \rho\cos\theta. \tag{22}$$

On substituting (20) and (21) into (16) and (17) with (22), we get for the coefficients of $f_1(G)$

$$D_\rho^4 H_1 = -\rho\sin\theta\left(\sin\theta\frac{\partial x_1}{\partial\rho}+\frac{\cos\theta}{\rho}\frac{\partial x_1}{\partial\theta}\right) \tag{23}$$

$$D_\rho^2 x_1 = \frac{P}{\rho\sin\theta}\left(\sin\theta\frac{\partial H_1}{\partial\rho}+\frac{\cos\theta}{\rho}\frac{\partial H_1}{\partial\theta}\right). \tag{24}$$

Equations (23) and (24) are transformed into cylindrical polar coordinates defined by $\xi = \rho\sin\theta$, $\eta = \rho\cos\theta$ as

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}\right)^2 H_1 = -\xi\frac{\partial x_1}{\partial\xi} \tag{25}$$

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial x_1}{\partial\xi}\right)+\frac{\partial^2 x_1}{\partial\eta^2} = \frac{P}{\xi}\frac{\partial H_1}{\partial\xi}. \tag{26}$$

The solution of (25) and (26) is obtained by Hankel transform. We define

$$\bar{x}_1 = \int_0^\infty \xi x_1 J_0(\lambda\xi)\,d\xi, \quad x_1 = \int_0^\infty \lambda\bar{x}_1 J_0(\lambda\xi)\,d\lambda, \tag{27}$$

$$\bar{H}_1 = \int_0^\infty H_1 J_1(\lambda\xi)\,d\xi, \quad H_1 = \int_0^\infty \lambda\xi\bar{H}_1 J_1(\lambda\xi)\,d\lambda. \tag{28}$$

Equations (25)–(28) give (see Singh [12])

$$x_1 = \sum_{n=1}^{3}\int_0^\infty \lambda D_n(\lambda)\exp(-\alpha_n\eta)J_0(\lambda\xi)\,d\lambda, \tag{29}$$

$$H_1 = \sum_{n=1}^{3}\int_0^\infty \lambda\xi E_n(\lambda)\exp(-\alpha_n\eta)J_1(\lambda\xi)\,d\lambda \tag{30}$$

where J_0 and J_1 are the Bessel functions of first kind and of order zero and unity respectively. D_n and E_n are constants related by

$$\lambda D_n(\lambda) = j_n^2(P\lambda^2)^{2/3}E_n(\lambda) \tag{31}$$

$$\alpha_n^2 = \lambda^2+j_n(P\lambda^2)^{1/3}, \quad n = 1,2,3 \tag{32}$$

and

$$(j_1,j_2,j_3) = \left(1, \frac{-1\pm\sqrt{3}i}{2}\right).$$

(iii) *Matching the Stokes and Oseen solutions*

On writing equation (10) in terms of Oseen variables defined by (15), its contribution to x is seen to be

$$x \sim A\left(\rho+\frac{\delta G^{3/2}}{\rho^2}\right)\cos\theta. \tag{33}$$

Comparing (33) with (20) and (22), we find

$$A = 1, \quad f_1(G) = G^{3/2}. \tag{34}$$

From (33) it follows that the constants $D_n(\lambda)$ should be such that when ξ and η tend to small values of order unity, $x_1 \sim \delta\cos\theta/\rho^2$. It is well known (Gradshteyn and Ryzhik [18])

$$\int_0^\infty \lambda e^{-\lambda\eta}J_0(\lambda\xi)\,d\lambda = \frac{\eta}{(\xi^2+\eta^2)^{3/2}} = \frac{\cos\theta}{\rho^2}. \tag{35}$$

If we assume

$$D_n(\lambda) = \delta/3, \tag{36}$$

it can be shown that the integral in (29) becomes a particular case of (35) and $x_1 \sim \delta\cos\theta/\rho^2$. To prove this, $\exp(-\alpha_n\eta)$ is expanded in a Taylor series such that $(\alpha_n\eta) \sim (\lambda\eta)$ as

$$\exp(-\alpha_n\eta) = \exp(-z)+(\alpha_n\eta-z)[\exp(-z)]' +(\alpha_n\eta-z)^2[\exp(-z)]''/2!+(\alpha_n\eta-z)^3[\exp(-z)]'''/3! +\dots, \tag{37}$$

where dashes denote differentiation with respect to z and $\lambda\eta = z$. Also

$$\alpha_n = \lambda\left\{1+\frac{(P\lambda^2)^{1/3}}{2\lambda^2}j_n-\frac{(P\lambda^2)^{2/3}}{8\lambda^4}j_n^2+\frac{P}{16\lambda^4}+\dots\right\} \tag{38}$$

$$\begin{aligned}\exp(-\alpha_n\eta) &= \exp(-z)+\left\{\frac{(P\lambda^2)^{1/3}}{2\lambda^2}j_n-\frac{(P\lambda^2)^{2/3}}{8\lambda^4}j_n^2+\frac{P}{16\lambda^4}\right\}z[\exp(-z)]'\\ &+\left\{\frac{(P\lambda^2)^{1/3}}{2\lambda^2}j_n-\frac{(P\lambda^2)^{2/3}}{8\lambda^4}j_n^2+\frac{P}{16\lambda^4}\right\}^2\times\frac{z^2}{2}[\exp(-z)]''+\left\{\frac{(P\lambda^2)^{1/3}}{2\lambda^2}j_n-\frac{(P\lambda^2)^{2/3}}{8\lambda^4}j_n^2\right.\\ &\left.+\frac{P}{16\lambda^4}\right\}^3\frac{z^3}{6}[\exp(-z)]'''+\dots\end{aligned} \tag{39}$$

Substituting (36) into (29) and using only the first term on the RHS of (39) (other terms will be considered later) and the result (35), we obtain

$$x_1 \simeq \int_0^\infty \delta\lambda\exp(-\lambda\eta)J_0(\lambda\xi)\,d\lambda = \delta\cos\theta/\rho^2. \tag{40}$$

Equations (36) and (31) give

$$E_n(\lambda) = \delta\lambda/3j_n^2(P\lambda^2)^{2/3}. \tag{41}$$

Dividing (39) by $j_n^2(P\lambda^2)^{2/3}$ and then summing it, we get

$$\sum_{n=1}^{3}\frac{\exp(-\alpha_n\eta)}{j_n^2(Pr\lambda^2)^{2/3}} = \frac{z}{8\lambda^4}\exp(-z)+\frac{z^2}{8\lambda^4}\exp(-z). \tag{42}$$

When (42) is substituted in (30) with E_n given by (41), it becomes

$$H_1 \simeq \int_0^\infty \delta\left[\frac{\xi\eta}{8\lambda}\exp(-\lambda\eta)+\frac{\xi\eta^2}{8}\exp(-\lambda\eta)\right]J_1(\lambda\xi)\,d\lambda$$

$$= \delta\frac{\xi\eta}{8}\left[\frac{(\xi^2+\eta^2)^{1/2}-\eta}{\xi}+\frac{\eta}{\xi}\left(1-\frac{\eta}{(\xi^2+\eta^2)^{1/2}}\right)\right]$$

$$= \delta\sin^2\theta\cos\theta\rho^2/8. \tag{43}$$

When equation (12) is expressed in terms of Oseen's variables defined by (15) and is compared with (43), we find

$$B_n = 0, \quad C_n = 0$$

and

$$\psi_1 = (\delta\sin^2\theta\cos\theta/8)(r-1/r)^2. \tag{44}$$

(iv) *Higher approximation*

Now that the Stokes solution T_0 (10) and ψ_1 (44) has been obtained, it will be shown how a knowledge of the Oseen solution x_1 (29) enables one to calculate the next higher Stokes approximation T_2. From (2), we get

$$\nabla^2 T_2 = -\frac{P}{r^2\sin\theta}\left[\frac{\partial\psi_1}{\partial r}\frac{\partial T_0}{\partial\theta}-\frac{\partial\psi_1}{\partial\theta}\frac{\partial T_0}{\partial r}\right], \tag{45}$$

whose solution, (only particular integral) satisfying (5) is

$$T_2(r,\theta) = \frac{P(5\cos^3\theta-3\cos\theta)}{240r^5}\times[r^7-3r^5+4r^4+3r^3-12r^2+10r-3]$$
$$+\frac{P\cos\theta}{240r^5}[-12r^7-24r^5+12r^4+4r^3+24r^2-4], \tag{46}$$

or

$$T_2(r,\theta) = \frac{P(5\cos^3\theta-3\cos\theta)}{-480r^5}\times[r^7-3r^5-2r^4+3r^3+6r^2-6.875r+1.5]$$
$$+\frac{P\cos\theta}{-480r^5}[-12r^7-24r^5-6r^4+4r^3-12r^2+2]. \tag{47}$$

Equation (46) is the solution satisfying $T_2 = 0$ at $r = 1$, i.e. for isothermal sphere and (47) satisfies $\partial T_2/\partial r = 0$ at $r = 1$, true for the thermally insulated sphere. If we can show that the remaining terms in equation (39) for x_1 represent the highest order terms in equation (46) and (47), i.e.

$$-\frac{\delta P(5\cos^3\theta-3\cos\theta)}{240}r^2+\frac{\delta P\cos\theta}{20}r^2 \tag{48}$$

then we are justified in neglecting the complementary part of the solution of (45) and (46) and (47) represents the next approximation. Equation (39) gives

$$\sum_{n=1}^{3}\exp(-\alpha_n\eta) = 3\exp(-\lambda\eta)$$
$$-\left[\frac{\eta}{16\lambda^3}+\frac{\eta^2}{16\lambda^2}+\frac{\eta^3}{48\lambda}\right]3\delta P\exp(-\lambda\eta)+\ldots. \tag{49}$$

The first term on the RHS of (49) has already been considered in (40). From (29) and (49), we obtain

$$x_1 \simeq -P\delta\int_0^\infty \lambda\left[\frac{\eta}{16\lambda^3}+\frac{\eta^2}{16\lambda^2}+\frac{\eta^3}{48\lambda}\right]\times\exp(-\lambda\eta)J_0(\lambda\xi)\,d\lambda. \tag{50}$$

The third integral on the RHS of equation (50) can be evaluated [see equation (35)]

$$\int_0^\infty \exp(-\lambda\eta)J_0(\lambda\xi)\,d\lambda = \frac{1}{(\xi^2+\eta^2)^{1/2}} = \frac{1}{\rho}. \tag{51}$$

The existence of the first two integrals of (49) is justified with the help of the theory of distributions and is evaluated according to the generalized Fourier transform technique (Lighthill [19]). Integrating by parts, we get

$$I = \int_{-\infty}^{\infty}\exp(-\lambda\eta)J_0(\lambda\xi)\frac{d\lambda}{\lambda^2}$$
$$= \left[-\frac{1}{\lambda}\exp(-\lambda\eta)J_0(\lambda\xi)\right]_{-\infty}^{\infty}$$
$$+\int_{-\infty}^{\infty}\frac{1}{\lambda}[-\eta\exp(-\lambda\eta)J_0(\lambda\xi)-\xi\exp(-\lambda\eta)J_1(\lambda\xi)]\,d\lambda. \tag{52}$$

From (52), it follows

$$\int_{-\infty}^{\infty}\left[\frac{1}{\lambda^2}+\frac{\eta}{\lambda}\right]\exp(-\lambda\eta)J_0(\lambda\xi)\,d\lambda = -\xi\int_{-\infty}^{\infty}\exp(-\lambda\eta)J_1(\lambda\xi)\frac{d\lambda}{\lambda}. \tag{53}$$

We know (See Gradshteyn and Ryzhik [18])

$$\int_0^\infty \exp(-\lambda\eta)J_1(\lambda\xi)\frac{d\lambda}{\lambda} = \frac{(\xi^2+\eta^2)^{1/2}-\eta}{\xi}. \tag{54}$$

With the help of equations (51)–(54), it follows considering the relevant terms

$$x_1 \simeq -(\delta\rho^2\cos^3\theta/48)+\delta\rho^2\cos\theta/16. \tag{55}$$

It is, thus, shown that subsequent terms can be accomodated satisfactorily within the scheme adopted.

4. DISCUSSION OF RESULTS

For the isothermal sphere ($\delta = -1$), the stream function ψ (44) and the velocity components v_r and v_θ become

$$\psi = -(G\sin^2\theta\cos\theta/8r^2)(r^2-1)^2, \tag{56}$$

$$v_r = -G(3\cos^2\theta-1)(1-1/r^2)^2/8, \tag{57}$$

$$v_\theta = -(G\sin\theta\cos\theta/4)(1-1/r^4). \tag{58}$$

From (57), it follows that at $\theta = 0$, v_r is negative and at $\theta = \pi/2$, v_r is positive. It changes sign at $\cos\theta = 1/\sqrt{3}$, i.e. at $\theta = 54.4°$. Thus for this case inflow takes place at the pole ($\theta = 0$) and changes into outflow at the equator ($\theta = \pi/2$). Streamlines are shown graphically in the first quadrant of Fig. 1. These are qualitatively similar to Fig. 6, $s = \infty$ of Eichhorn *et al.* [11].

The temperature distribution is given by the equations (10) and (46) ($A = 1$, $\delta = -1$). Isotherms are plotted in Fig. 2.

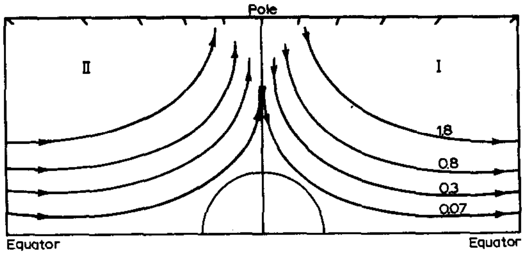

FIG. 1. 1.Streamlines in an axial plane (i) for isothermal sphere in quadrant I and (ii) for thermally insulated sphere in quadrant II.

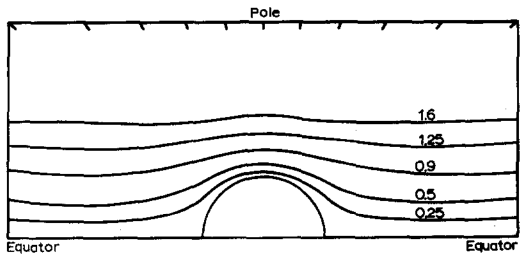

FIG. 2. Isotherms for the case of isothermal sphere $PG^2 = 0.5625$.

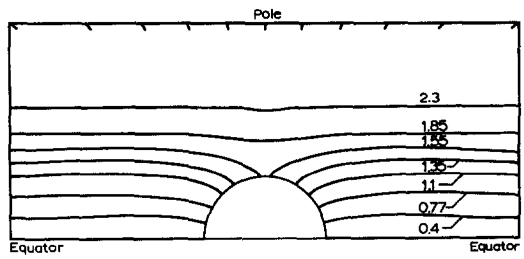

FIG. 3. Isotherms for the case of thermally insulated sphere $PG^2 = 0.5625$.

In this case since the upper hemispherical surface ($\theta = 0$ to $\theta = \pi/2$) is at a lower temperature than the surrounding fluid, heat is transfered from the fluid to the sphere. The amount of heat absorbed by the upper hemisphere is given by

$$Q_u = 2k\pi a^2 \int_{\theta=0}^{\theta=\pi/2} \sin\theta \left(\frac{\partial T'}{\partial r'}\right)_{r'=a} \mathrm{d}\theta \tag{59}$$

$$= k\pi a^2 \left(\frac{\mathrm{d}T_\infty}{\mathrm{d}y'}\right)\left[3 - \frac{13PG^2}{32}\right]. \tag{60}$$

Similarly we can calculate the amount of heat rejected by the lower hemisphere Q_l. The RHS of (59) when integrated within the limits from $\theta = \pi/2$ to $\theta = \pi$ gives $Q_l = -k\pi a^2.(\partial T_\infty/2y')(3\text{–}13PG^2/32)$. Thus we find $Q_l = -Q_u$, i.e. the sphere absorbs heat at the top and rejects it at the bottom in equal amount, as observed by Eichhorn *et al.* [11].

In the case of the thermally insulated sphere ($\delta = 0.5$) the stream function and velocity components are just (-0.5) times equations (56), (67) and (58). Hence all qualitative conclusions from the earlier equations also apply to this case but with senses reversed. Streamlines are sketched in quadrant II of Fig. 1 and these are similar to the flow about a stationary sphere in a rotating fluid (Singh [12]). The isotherms for this case are plotted in Fig. 3. Since nature of the flow lines as shown in Fig. 1 are in agreement with those for the flow due to a sphere rotating in a fluid at rest and the flow about a stationary sphere in a rotating fluid, we conclude that stratified and rotating flows are analogous (see Yih [13] and Veronis [14]).

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CONVECTION LIBRE D'UNE SPHERE DANS UN FLUIDE LEGEREMENT STRATIFIE THERMIQUEMENT

Résumé—On considère l'écoulement permanent et axisymétrique d'un fluide visqueux et stratifié verticalement autour d'une sphère fixe, dans un champ de gravité uniforme. On obtient des solutions analytiques par la technique des perturbations, valable pour de petits nombres de Grashof (modifiés). Deux cas sont considérés selon que la sphère soit athermane ou que la température de sa surface soit constante et égale à celle du fluide situé au plan diamétral. Les lignes de courant sont représentées graphiquement dans un plan axial, à proximité de la sphère. On trouve que pour la sphère athermane, l'écoulement est semblable à celui autour d'une sphère fixe dans un fluide en rotation: écoulement rentrant près de l'équateur et écoulement fuyant aux poles, la transition apparaissant à un angle de 34,5° à partir des poles. Pour la sphère isotherme, les lignes de courant sont semblables à celles d'un fluide au repos autour d'une sphère tournante.

FREIE KONVEKTION UM EINE KUGEL IN EINEM THERMISCH SCHWACH GESCHICHTETEN FLUID

Zusammenfassung—Es wird die stationäre, achsensymmetrische Strömung eines vertikal geschichteten zähen Fluides um eine feststehende Kugel in einem einheitlichen Gravitationsfeld betrachtet. Analytische Lösungen werden mit Hilfe der singulären Perturbationstechnik erhalten, welche für kleine (modifizierte) Grashof-Zahlen gültig ist. Es werden zwei Fälle betrachtet, die adiabate Kugel und die Kugel konstanter Oberflächentemperatur (entsprechend der Fluidtemperatur in Äquatorebene). Die Stromlinien der kugelnahen Strömung werden in einer axialen Ebene grafisch dargestellt. Die Strömung um die adiabate Kugel entsprach derjenigen eines rotierenden Fluides um eine stationäre Kugel mit Zuströmung am Äquator und Abströmung an den Polen; der Umschlag lag bei einem vom Pol aus gemessenen Winkel von 54,5°. Für den Fall der isothermen Kugel sind die Stromlinien ähnlich denjenigen um eine gleichförmig rotierende Kugel in einem ruhenden Fluid.

СВОБОДНАЯ КОНВЕКЦИЯ ОТ СФЕРЫ, НАХОДЯЩЕЙСЯ В ТЕРМИЧЕСКИ СЛАБО СТРАТИФИЦИРОВАННОЙ ЖИДКОСТИ

Аннотация — Рассматривается стационарное осесимметричное течение стратифицированной по вертикали вязкой жидкости над сферой в однородном поле тяжести. С помощью метода сингулярных возмущений, справедливого при небольших (модифицированных) числах Грасгофа, получены аналитические решения. Рассматриваются два случая: когда сфера термически изолирована и когда температура ее поверхности поддерживается постоянной и равной температуре жидкости в диаметральной плоскости. Представлены линии тока в осевой плоскости вблизи сферы. Найдено, что в случае термически изолированной сферы течение подобно потоку, омывающему неподвижную сферу во вращающейся жидкости, т. е. приток у экватора превращается в отток у полюсов, причем этот переход происходит под углом 54,5°, отсчитываемым от полюсов. Для изотермической сферы линии тока подобны линиям тока для сферы, равномерно вращающейся в покоящейся жидкости.